

# Casimir effect in dielectrics: Surface area contribution

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## ABSTRACT

In this paper we take a deeper look at the technically elementary but physically robust viewpoint in which the Casimir energy in dielectric media is interpreted as the change in the total zero point energy of the electromagnetic vacuum summed over all states. Extending results presented in previous papers [hep-th/9609195; hep-th/9702007] we approximate the sum over states by an integral over the density of states including finite volume corrections. For an arbitrarily-shaped finite dielectric, the first finite-volume correction to the density of states is shown to be proportional to the surface area of the dielectric interface and is explicitly evaluated as a function of the permeability and permittivity. Since these calculations are founded in an elementary and straightforward way on the underlying physics of the Casimir effect they serve as an important consistency check on field-theoretic calculations. As a concrete example we discuss Schwinger's suggestion that the Casimir effect might be the underlying physical basis behind *sonoluminescence*. The recent controversy concerning the relative importance of volume and surface contributions is discussed. For sufficiently large bubbles the volume effect is always dominant. Furthermore we can explicitly calculate the surface area contribution as a function of refractive index.

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## I. INTRODUCTION

The Casimir effect in dielectrics is the subject of intense on-going interest. While there is no doubt that the effect is real, certain suggested applications are somewhat controversial. For instance: Schwinger has suggested that the Casimir effect might be the underlying physics behind sonoluminescence [1–3], while Carlson, Goldman, and Pérez-Mercader have suggested possible applications to Gamma Ray Bursts [4]. More generally, the Casimir energy has sometimes been invoked as a possible driving mechanism for ultra-high-energy astrophysical processes such as quasars. We feel that all aspects of the discussion could benefit from the improved understanding of the basic physics we provide in this paper.

Historically, the techniques used to investigate the Casimir effect were typically a varied mixture of Schwinger's source theory, explicit calculations of electromagnetic Green functions (seasoned with time-splitting regularization), and sometimes, more physically based regulator schemes that take advantage of the analyticity properties of the frequency dependent refractive index.

A key early paper is that by Schwinger, de Raad and Milton [5].

Schwinger's most developed point of view can be gleaned from the series of papers he recently wrote wherein he explored the possible relevance of the Casimir effect to sonoluminescence [1–3]. For the evolution of his views on this subject see [6–9].

Schwinger found [1] that (for each polarization state) the “dielectric energy, relative to the zero energy of the vacuum, [is given] by

$$E = -V \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{1}{2} [\hbar c] k \left( 1 - \frac{1}{\sqrt{\epsilon}} \right). \quad (1)$$

This result can be interpreted in a straightforward manner as the integral of the *difference* in dispersion relations over the density of states [10,11].

In addition to Schwinger's bulk volume term, calculations by Milton *et al.* [12–14] indicate the existence of a surface correction. For a dilute (that is,  $\epsilon \approx 1$ ) spherical intrusion of radius  $R$  and dielectric constant  $\epsilon_1$  in a dilute dielectric medium of dielectric constant  $\epsilon_2$  ( $\epsilon_2 \approx 1$ ), with the eigenmode sum regulated by time-splitting, the surface contribution is equivalent to<sup>1</sup>

$$E_{\text{surface}} \approx -\frac{1}{4} \hbar c (\epsilon_1 - \epsilon_2)^2 R^2 \frac{1}{(c\tau)^3}. \quad (2)$$

A controversy has recently arisen over whether or not Schwinger's volume term should be retained, and whether or not the surface term is the leading term in the Casimir energy [13,14]. We have shown elsewhere [10,11] that the presence of the volume term is generic, and have (among other arguments) adduced reasons based on density-of-states calculations to bolster Schwinger's calculation. In this paper we shall pursue this matter further and shall extract as much physics as possible from these density-of-states calculations.

The discussion, though elementary from a technical perspective, is quite sufficient to give the most important

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<sup>1</sup>See equation (51) of [13], equation (7.5) of [14], or the equivalent equation (41) of [12]. Those calculations only deal with spherical dielectric balls with frequency independent dielectric properties, and use an explicit time-splitting regularization. The numerical coefficient in this surface term is regularization dependent and it does not appear to be possible to relate its absolute normalization to the number we will calculate using Schwinger's wave-number cutoff.

dominant contributions to the Casimir energy. These results serve as an important consistency check on more sophisticated field-theoretic calculations.

Furthermore, the present analysis extends Schwinger's result by verifying that generically surface terms do in fact show up, but as *sub-dominant* corrections to the dominant volume contribution. General arguments of this type are particularly useful because they allow us to study *arbitrary* shapes and not be limited by requirements of spherical symmetry.

We first discuss some general properties of the bulk volume term, noting in particular the dependence upon a physically meaningful ultraviolet cutoff, and then turn to the issue of finite volume effects. While finite volume effects in conductors (or more precisely, for Dirichlet, Neumann, and Robin *boundary* conditions) are well understood, the analogous problem for dielectric *junction* conditions (or even acoustic junction conditions) is considerably less clear cut. We attack the problem of finite volume effects in the presence of junction conditions via an extension of the Balian-Bloch analysis for boundary conditions [15,16]. We show that the presence of a dielectric interface modifies the density of states by a term proportional to the surface area of the interface and calculate the proportionality constant as an explicit function of the dielectric permittivity and permeability. (For the related, and simpler, acoustic interface the change in density of states is related to the physical fluid densities on the two sides of the interface.)

Finally, we apply this formalism to the estimation of the (electromagnetic) Casimir energy in generic dielectrics. We show that for dielectric bubbles large compared to the cutoff wavelength the volume term is dominant. We point out that the numerical value of the net Casimir energy is strongly dependent on the details of the high frequency cutoff. Within the context of sonoluminescence this high-frequency sensitivity might explain the fact that small admixtures of gas in the bubble undergoing sonoluminescence can have large effects on the total energy radiated: a small resonance in the medium-frequency behaviour of the refractive index can be magnified by phase space effects, and lead to dramatic changes in the total energy budget.

We mention in passing that there will also be an acoustic Casimir energy associated with the phonon modes [17]. The acoustic Casimir energy (while always present) is numerically negligible in comparison to the electromagnetic effect being suppressed by a factor of (speed of sound/speed of light).

## II. THE DENSITY OF STATES: BULK TERM

The physics underlying the Casimir effect is that every eigenmode of the photon field has zero point energy  $E_n = (1/2)\hbar\omega_n$ ; the Casimir energy is the difference in zero point energies between any two well defined physical

situations

$$E_{Casimir}(A | B) = \sum_n \frac{1}{2} \hbar [\omega_n(A) - \omega_n(B)]. \quad (3)$$

We always need a regulator to make sense of this energy difference, though in many cases of physical interest (such as dielectrics) the physics of the problem will automatically regulate the difference for us and make the results finite. Adding over all eigenmodes is prohibitively difficult, so it is in general more productive to replace the sum over states by an integral over the density of states.

Suppose we have a finite volume  $V$  of some bulk dielectric in which the dispersion relation for photons is given by some function  $\omega_1(k)$ , which describes the photon frequency as a function of the wave-number (three-momentum)  $k$ . Suppose this dielectric to be embedded in an infinite background with different dielectric properties described by a different dispersion relation  $\omega_2(k)$ . We regulate infra-red divergences by putting the whole universe in a box of finite volume  $V_\infty$ , and calculate the bulk contribution to the total zero-point energy of the electromagnetic field by summing the photon energies over all momenta (and polarizations), using the usual and elementary density of states:  $[\text{Volume}] d^3\vec{k}/(2\pi)^3$ . (In the next section we shall look at finite-volume corrections to this density of states.)

Including photon modes both inside and outside the dielectric body the energy of the system is

$$E_{embedded-body} = 2V \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2} \hbar \omega_1(k) + 2(V_\infty - V) \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2} \hbar \omega_2(k). \quad (4)$$

Note that outside the dielectric body the photon dispersion relation is that of the embedding dielectric  $\omega_2(k)$ . Note also that we shall always use the subscript 2 to refer to the region *outside* the embedded body, and shall use the subscript 1 to refer to the region *inside*.

If the embedded body is removed, and the hole simply filled in with the embedding medium, we can calculate the total zero-point energy as

$$E_{homogeneous} = 2V_\infty \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2} \hbar \omega_2(k). \quad (5)$$

We *define* the Casimir energy by subtracting these two zero-point energies [10,11]

$$E_{Casimir} \equiv E_{embedded-body} - E_{homogeneous} = 2V \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2} \hbar [\omega_1(k) - \omega_2(k)]. \quad (6)$$

The physical import of this definition is clear: The Casimir energy is defined as the *change* in the zero-point energy due to a change in the medium.

Note also that the physical meaning of the zero of energy is clear: the zero of energy is here taken to be that corresponding to a homogeneous dielectric with dispersion relation  $\omega_2(k)$ .

To be obtuse, we could use a different zero for the energy — this makes no difference as long as we keep the same zero throughout any particular calculation. For instance, the zero-point energy of the Minkowski vacuum is

$$E_{Minkowski} = 2V_\infty \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2} \hbar c k. \quad (7)$$

Thus an alternative *definition* for the Casimir energy is then

$$\begin{aligned} E_{Casimir}^{alternative} &\equiv E_{embedded-body} - E_{Minkowski} \\ &= 2V \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2} \hbar [\omega_1(k) - ck] \\ &\quad + 2(V_\infty - V) \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2} \hbar [\omega_2(k) - ck]. \end{aligned} \quad (8)$$

For this alternative definition, the zero of energy is clearly the Minkowski vacuum. As long as you stick with one fixed definition throughout the calculation, or better yet, calculate Casimir energy differences directly, quibbling about the zero of energy does not matter. (Of course, if you change the zero of energy in the middle of the calculation the answers will be meaningless.)

From the general considerations in [10,11] we know that the integrand must go to zero at large wave-number, and in fact, for any pair of real physical dielectrics the integrand must go to zero sufficiently rapidly to make the integral converge.

An integration by parts yields

$$\begin{aligned} E_{Casimir} &= \frac{V\hbar}{6\pi^2} \int_0^\infty d(k^3) [\omega_1(k) - \omega_2(k)] \\ &= \frac{V\hbar}{6\pi^2} \left[ (k^3 [\omega_1(k) - \omega_2(k)]) \Big|_0^\infty \right. \\ &\quad \left. - \int_0^\infty [d\omega_1(k) - d\omega_2(k)] k^3 \right]. \end{aligned} \quad (9)$$

The boundary term vanishes because of the asymptotic behaviour of the  $\omega_i(k)$ . The substitution  $k = \omega_i(k)n_i$  then yields

$$E_{Casimir} = + \frac{V\hbar}{6\pi^2 c^3} \int_0^\infty \omega^3 [n_2^3(\omega) - n_1^3(\omega)] d\omega. \quad (10)$$

While the difference between the refractive indices in the above expression goes to zero sufficiently rapidly to make the integral converge, it must be noted that the prefactor of  $\omega^3$  implies that the net Casimir energy will be relatively sensitive to the high frequency behaviour of the refractive indices.

If the Casimir effect ultimately proves to be the correct physical explanation for sonoluminescence, this sensitivity to the details of the refractive index might plausibly explain why sonoluminescence is sensitive to the admixture of small trace gases into the bubble. (Of course the present calculation is static, but the energy calculated in this way will be the maximum energy that could possibly be released in a more realistic dynamical calculation.) To make this qualitative statement quantitative we would need a detailed model for the refractive index as a function of frequency—a task that is beyond the scope of this paper.

### III. THE DENSITY OF STATES: FINITE-VOLUME EFFECTS

#### A. Generalities

We now look at the contribution arising from subdominant finite-volume corrections to the density of states. The key point here is that the existence of finite-volume terms proportional to the surface area of the dielectric is a *generic* result. The fact that previous calculations [12–14] encountered a surface tension term proportional to (cutoff)<sup>3</sup> is hereby explained on general physical grounds without recourse to special function theory.

We must notice at this stage that the dominant contribution to the Casimir energy is proportional to volume, as the canonical bulk expression for the density of states is proportional to the volume: [Volume]  $d^3\vec{k}/(2\pi)^3$ . It is reasonably well-known, though perhaps not so elementary, that for fields subject to *boundary* conditions (Dirichlet, Neumann, Robin) the density of states is in general modified by finite volume effects. In this paper we wish to extend these ideas to fields subject to *junction* conditions (acoustic, dielectric).

For boundary conditions the general result is

$$\sum_n \sim V \int \frac{d^3\vec{k}}{(2\pi)^3} + S \int \xi \frac{d^3\vec{k}}{(2\pi)^3 k} + \dots \quad (11)$$

These are the first two terms in an asymptotic expansion in  $1/k$ . For Dirichlet, Neumann, and Robin boundary conditions the coefficients can be related directly to the known asymptotic behaviour of the Heat Kernel—they are simply the Seeley–DeWitt coefficients in disguise and can be obtained, for instance, by suitably transforming the results presented in the monograph by Gilkey [19].

There are additional terms in this expansion, proportional to the various monomials appearing in the general formulae for the higher Seeley–DeWitt coefficients, but we do not further address this issue here except to point out that the next term is proportional to the integral of the trace of the extrinsic curvature over the boundary.

An elementary discussion of the general existence of such terms can be found in the textbook by Pathria [18],

while a more extensive treatment can be found in the papers by Balian and Bloch [15,16].

For Dirichlet, Neumann, and Robin boundary conditions the dimensionless variable  $\xi$  is a known function of the boundary conditions imposed.

If we let  $N(k)$  denote the number of eigenmodes with wave-number less than  $k$ , then from the above we can write

$$N(k) \sim \frac{1}{2\pi^2} \left( \frac{1}{3} V k^3 + \frac{1}{2} \xi S k^2 + O[k] \right). \quad (12)$$

We shall now perform the analogous analysis for junction conditions, adapting the Balian–Bloch formalism as needed. Our formalism is applicable to both boundary conditions and junction conditions. For clarity, and to aid in consistency checking, we carry out brief parallel computations for the boundary condition case.

## B. Scalar field

We start for simplicity with a scalar, rather than electromagnetic, field. We are interested in the following eigenvalue problem

$$\Delta\phi + k^2\phi = 0; \quad B[\phi] = 0. \quad (13)$$

Here  $B[\phi]$  denotes the boundary conditions imposed. Common boundary conditions are tabulated below.

### Dirichlet boundary conditions:

( $\phi = 0$  on the boundary)

$$\xi = -\pi/4. \quad (14)$$

### Neumann boundary conditions:

( $\partial_n\phi = 0$  on the boundary; where  $\partial_n$  denotes the normal derivative)

$$\xi = +\pi/4. \quad (15)$$

### Robin boundary conditions:

( $\partial_n\phi = \kappa\phi$  on the boundary;  $\kappa$  real)

$$\xi = +\pi/4. \quad (16)$$

### Surface damped boundary conditions:

( $\partial_n\phi = \kappa\phi$  on the boundary;  $\kappa$  real; note that the eigenvalue is now explicitly present in the boundary condition as well as in the differential equation)

$$\xi = \frac{\pi}{4} - \frac{1}{2} \text{Im} \left[ \ln \left( \frac{1+i\kappa}{1-i\kappa} \right) \right] = \frac{\pi}{4} - \arctan(\kappa). \quad (17)$$

These results can be read off, for instance, from the paper by Balian and Bloch [15].

Comparing the Robin and surface damped boundary conditions, it might naively be tempting to write

$$\xi_{Robin}(\kappa) = \xi_{damped}(\kappa/k). \quad (18)$$

However in the present context—an asymptotic expansion in  $1/k$ —such an expression is meaningless. The best we can do is to say that

$$\xi_{Robin}(\kappa) = \lim_{k \rightarrow \infty} \xi_{damped}(\kappa/k). \quad (19)$$

Thus for Robin boundary conditions we keep only the dominant  $k \rightarrow \infty$  piece of the Balian–Bloch result.

On the other hand, in the surface damped boundary condition (because of the explicit factor of  $k$  appearing in this boundary condition) it is meaningful to keep the inverse tangent term of the Balian–Bloch result in our expression for  $\xi$ . (As a consistency check, these coefficients are also calculated as special cases of the general formalism we shall develop below.)

### Acoustic junction conditions:

We are ultimately interested in *junction conditions*, rather than *boundary conditions*. For definiteness, we can think of an acoustic junction, wherein an acoustic wave propagates across some fluid interface: say a bubble of some dense fluid embedded in a lighter fluid. In terms of the densities of the fluids,  $(\rho_1, \rho_2)$ , and the velocity potentials,  $(\phi_1, \phi_2)$ , the acoustic junction conditions are

$$\rho_1\phi_1 = \rho_2\phi_2, \quad (20)$$

$$\partial_n\phi_1 = \partial_n\phi_2. \quad (21)$$

(See [20, page 24] or [20, page 81]. These two conditions represent, respectively, the continuity of the pressure and the normal component of the velocity at the interface.) We must point out at this stage that the change in propagation speed and/or density causes a certain amount of reflection and refraction, which then changes the density of states in the fluid both *inside* and *outside* the bubble (*i.e.* on both sides of the interface) according to the general scheme

$$\sum_{inside} \sim V \int \frac{d^3\vec{k}}{(2\pi)^3} + S \int \xi_{in} \frac{d^3\vec{k}}{(2\pi)^3 k} + \dots \quad (22)$$

$$\sum_{outside} \sim (V_\infty - V) \int \frac{d^3\vec{k}}{(2\pi)^3} + S \int \xi_{out} \frac{d^3\vec{k}}{(2\pi)^3 k} + \dots \quad (23)$$

For the case of acoustic junction conditions, the dimensionless variables  $\xi_{out/in}$  have not yet been calculated. We present the calculation below, for now merely quoting the final result:

$$\xi_{out}(\rho_1, \rho_2) = \frac{\pi}{4} \left[ \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} \right] = -\xi_{in}(\rho_1, \rho_2). \quad (24)$$

### Formulation of the problem:

On general grounds, we expect the  $\xi$  to be a function of both the acoustic refractive index (that is, a function of the relative acoustic velocities), and the relative densities.

If  $v_0$  is some arbitrary reference speed, we can define the refractive indices by

$$n_1 \equiv k_1 v_0 / \omega \quad \text{and} \quad n_2 \equiv k_2 v_0 / \omega, \quad (25)$$

and further define the relative refractive index by  $n = n_1 / n_2$ . (Note in particular that  $\omega$  is continuous across the interface, whereas  $k_i$  is not.) It is also useful to define the density contrast by  $\rho = \rho_1 / \rho_2$ .

In the special case where there is no dispersion, the phase and group velocities are equal and we simply have

$$n_1 = v_0 / v_1 \quad \text{and} \quad n_2 = v_0 / v_2. \quad (26)$$

We know, from first principles, that as  $n \rightarrow 1$  and  $\rho \rightarrow 1$  the acoustic boundary becomes indistinguishable, as both fluids have the same density and refractive index, so we must have

$$\xi_{out/in}(n, \rho) \rightarrow 0 \quad \text{as} \quad n \rightarrow 1 \quad \text{and} \quad \rho \rightarrow 1. \quad (27)$$

To calculate  $\xi(n, \rho)$  for acoustic junction conditions, we modify the discussion of Balian and Bloch [15, page 407] to derive an expression for  $\xi(n, \rho)$  in terms of an integral involving the reflection coefficient  $R(\rho, n; \vec{k})$ .

We start from the result for the density of states in terms of the time-independent Green function [15, equation (II.6), page 409]. Taking  $N(k)$  to be the number of modes with wave-number less than  $k$ , we can construct a suitably smoothed density of states formally described by the relation

$$\rho_{dos}(k) = \left[ \frac{dN}{dk} \right]_{smoothed}. \quad (28)$$

(Details of the smoothing procedure can be found in [15].) Note that we prefer to express the density of states in terms of the wave-number  $k_i$  rather than in terms of the variable  $E = k_i^2$ . See equation (I.3) on page 402 of [15]. Thus

$$\rho_{dos}(k) \sim \frac{dN}{dk} \sim \frac{dE}{dk} \frac{dN}{dE} \sim 2k \rho_{dos}^{BB}(E). \quad (29)$$

In terms of the asymptotic expansion of interest

$$\rho_{dos}(k) \sim \frac{1}{2\pi^2} (V k^2 + S \xi k + O[1]). \quad (30)$$

Working on either side of the interface (with  $i$  taking on the values “in” or “out” as appropriate) equation (II.6) on page 409 of [15] yields

$$\rho_{dos}^i(k_i) = \frac{2k_i}{\pi} \int d^3 \vec{x} \lim_{\vec{x}' \rightarrow \vec{x}} \text{Im}[G(\vec{x}, \vec{x}'; k_i + i\epsilon)], \quad (31)$$

where the integration over  $x$  now runs only over region  $i$  as appropriate.

It is important to realize that the Balian–Bloch formalism is built up under the assumption that all the eigenvalues are real—this constrains the type of problems we can deal with to loss-free undamped situations.

We are interested in an arbitrary interface, but provided the interface is smooth, we can locally replace it by its tangent plane. This approximation is equivalent to neglecting sub-dominant pieces proportional to the trace of the extrinsic curvature. (If we were interested in explicitly calculating the next coefficient in the expansion we would have to locally approximate the surface by its osculating ellipsoid, as done for the case of boundary conditions by Balian and Bloch.)

Truncating the expansion at the surface area term, we locally approximate the interface by a plane interface, located at  $z = 0$ , with region 2 (the outside) at  $z > 0$  and region 1 (the inside) at  $z < 0$ . We are only interested in the diagonal part of the Green function. To calculate this diagonal part in region  $i$  we can assume the source is also in region  $i$  and write the total Green function in this region as a sum of a *direct* and a *reflected* contribution.

The direct part of the Green function is responsible for the bulk contribution to the density of states, while the reflected part of the Green function gives the surface contribution. Since, in the tangent plane approximation, we are dealing with a perfectly flat interface higher order contributions are explicitly excluded.

The volume contribution has already been calculated in [10,11], and we are now interested in the extra piece of the Green function that arises from reflection at the interface. Using cylindrical coordinates, the contribution to the Green function due to the reflected wave can be put into the Sommerfeld representation (an integral over transverse wave-number  $k_t$ )

$$\begin{aligned} G_{reflection}^i(\vec{x}, \vec{x}'; k_i) &= \frac{i}{4\pi} \int_0^\infty R^i(k_i, k_t) J_0(k_t r) \\ &\quad \times \frac{\exp[iK(k_i, k_t)(z + z')]}{K(k_i, k_t)} k_t dk_t. \end{aligned} \quad (32)$$

(See equation (4.2.5) on page 103 of [20], with an appropriate change of notation.) Note that  $R(k_i, k_t)$  is the reflection coefficient. It is a function of the frequency and the transverse wave-number and will consequently depend on the precise nature of the boundary conditions imposed. The Sommerfeld representation has the interesting feature that it expresses a Green function, which is related to the behaviour of spherical waves, in terms of a reflection coefficient defined for plane waves. Here

$$K(k_i, k_t) = \sqrt{k_i^2 - k_t^2}. \quad (33)$$

More explicitly

$$K_{out}(k_2, k_t) = \sqrt{k_2^2 - k_t^2}. \quad (34)$$

$$K_{in}(k_1, k_t) = \sqrt{k_1^2 - k_t^2}. \quad (35)$$

If we look at the diagonal part of this reflection contribution, ( $\vec{x} = \vec{x}'$ ), and note that  $J_0(0) = 1$  we immediately see

$$G_{\text{reflection}}^i(\vec{x}, \vec{x}; k_i) = \frac{i}{4\pi} \int_0^\infty \frac{R^i(k_i, k_t) \exp[2iK(k_i, k_t)z]}{K(k_i, k_t)} k_t dk_t. \quad (36)$$

(Note that we are calculating what is in field theory parlance an off-shell Green function. The integration over  $k_t$  is an integration over all off-shell transverse momenta, and this integration is not to be limited by any on-shell constraint such as  $k_t \leq k_i$ .)

For the density of states (counting only the appropriate contribution arising from either side of the interface, that is,  $z > 0$  or  $z < 0$ )

$$\begin{aligned} \rho_{\text{reflection}}^i(k_i) &= \frac{ik_i}{2\pi^2} S \int_0^\infty dz \operatorname{Im} \left[ \int_0^\infty k_t dk_t R^i(k_i + i\epsilon, k_t) \right. \\ &\quad \times \left. \frac{\exp[2iK(k_i + i\epsilon, k_t)z]}{K(k_i + i\epsilon, k_t)} \right]. \end{aligned} \quad (37)$$

The  $z$  integration is trivial. (Because  $k_i$  has a small positive imaginary part, which is inherited by  $K(k_i, k_t)$ , we can guarantee convergence of this integral.)

$$\rho_{\text{reflection}}^i(k_i) = -\frac{k_i}{4\pi^2} S \operatorname{Im} \left[ \int_0^\infty \frac{R^i(k_i + i\epsilon, k_t)}{K(k_i + i\epsilon, k_t)^2} k_t dk_t \right]. \quad (38)$$

It is useful to define the dimensionless variable  $u = k_t/(k_i + i\epsilon)$ , so that  $u$  has a small *negative* imaginary part. We get

$$\rho_{\text{reflection}}^i(k_i) = -\frac{k_i}{4\pi^2} S \operatorname{Im} \left[ \int_0^\infty \frac{R^i(k_i, u - i\epsilon)}{1 - (u - i\epsilon)^2} u du \right]. \quad (39)$$

If we now take this contribution to the quantity  $\rho_{\text{dos}}$ , and convert to the  $\xi^i$  variable as defined in this paper using

$$\xi^i = \frac{2\pi^2}{kS} \rho_{\text{reflection}}^i, \quad (40)$$

we find

$$\xi^i(k_i) = -\frac{1}{2} \operatorname{Im} \left[ \int_0^\infty \frac{R^i(k_i + i\epsilon, k_t)}{K(k_i + i\epsilon, k_t)^2} k_t dk_t \right]. \quad (41)$$

Equivalently

$$\xi^i(k_i) = -\frac{1}{2} \operatorname{Im} \left[ \int_0^\infty \frac{R^i(k_i, u - i\epsilon)}{1 - (u - i\epsilon)^2} u du \right]. \quad (42)$$

This is our general result for the surface contribution to the density of states. The surface term is seen to be a

suitable average of the reflection coefficient appropriate to the boundary conditions at hand. (Note that if we were on-shell, we would interpret  $u$  as the sine of the angle of incidence, and  $u$  would then be limited to the range  $u \in [0, 1]$ . As this is an off-shell computation for the off-shell Green function, the range of integration goes all the way to infinity and trying to interpret  $u$  as the sine of the angle of incidence only leads to unnecessary confusion. Indeed, in calculating this Green function, we are effectively dealing with a spherical incident wave, so there are many angles of incidence  $\theta_i$ . To identify  $u$  as the sine of *the* angle of incidence only makes sense for an incident plane wave, and is in the present context meaningless.)

The application of this result to specific cases of interest merely requires us to calculate the relevant reflection coefficients and perform the integrations.

### The integral for standard boundary conditions:

In some well known cases the relevant integrations are straightforward. For example for Dirichlet, Neumann, and Robin boundary conditions the reflection coefficients are  $-1$ ,  $+1$ , and  $+1$  respectively, and integrating out to some large cutoff value of  $u$  we have

$$\begin{aligned} &\int_0^U \frac{1}{1 - (u - i\epsilon)^2} u du \\ &= \frac{1}{2} \int_0^{U^2} \frac{1}{1 - (x - i\epsilon)} dx \\ &= \frac{1}{2} [-\ln\{1 - (x - i\epsilon)\}]_0^{U^2} \\ &= \frac{1}{2} (\ln\{1 + i\epsilon\} - \ln\{1 - U^2 + i\epsilon\}) \\ &= -[i\pi + \ln(U^2 - 1)]/2 \\ &\approx -\frac{i\pi}{2} - \ln(U). \end{aligned} \quad (43)$$

Note that the integral itself diverges, though the imaginary part is both finite and independent of the cutoff. Taking this imaginary part gives

$$\xi = \mp \frac{\pi}{4}. \quad (44)$$

This reproduces the standard results quoted above. [Equations (14–16).]

The surface damped boundary condition is a little trickier. In this case the reflection coefficient can be shown to be

$$R(u) = \frac{\sqrt{1 - u^2} - i\kappa}{\sqrt{1 - u^2} + i\kappa}. \quad (45)$$

(See, for example, equations (3.4.4) and (3.4.5) on page 87 of DeSanto [20], and translate to our notation. Note that an analytic continuation in  $\kappa$  is required to turn the surface impedance boundary condition discussed there

into the surface damped boundary condition discussed here.)

Subtracting and adding 1 to the integrand converts the integral into that encountered in the previous calculation plus an integral that is well-behaved at infinity. The relevant integral is again elementary:

$$\int_0^\infty \frac{1}{1-u^2} \left[ \frac{\sqrt{1-u^2}-i\kappa}{\sqrt{1-u^2}+i\kappa} - 1 \right] u du = +2 \ln(1+i\kappa). \quad (46)$$

Taking the imaginary part of the above reproduces the result announced in equation (17):

$$\xi = \frac{\pi}{4} - \arctan(\kappa). \quad (47)$$

Checking the above:

$$\begin{aligned} & \int_0^\infty \frac{1}{1-(u-i\epsilon)^2} \left[ \frac{\sqrt{1-(u-i\epsilon)^2}-i\kappa}{\sqrt{1-(u-i\epsilon)^2}+i\kappa} - 1 \right] u du \\ &= \int_0^\infty \frac{1}{1-(u^2-i\epsilon)} \left[ \frac{\sqrt{1-(u^2-i\epsilon)}-i\kappa}{\sqrt{1-(u^2-i\epsilon)}+i\kappa} - 1 \right] u du \\ &= \int_0^\infty \frac{1}{1-(u^2-i\epsilon)} \left[ \frac{-2i\kappa}{\sqrt{1-(u^2-i\epsilon)}+i\kappa} \right] u du \\ &= \int_0^\infty \frac{1}{1-u'+i\epsilon} \left[ \frac{-i\kappa}{\sqrt{1-u'+i\epsilon}+i\kappa} \right] du' \\ &= \int_{-1}^\infty \frac{1}{u''-i\epsilon} \left[ \frac{i\kappa}{\sqrt{-u''+i\epsilon}+i\kappa} \right] du'' \\ &= \int_{-1}^\infty \frac{1}{u''-i\epsilon} \left[ \frac{\kappa}{\sqrt{u''-i\epsilon}+\kappa} \right] du'' \\ &= \int_{-i}^\infty \frac{2}{\tilde{u}-i\epsilon} \left[ \frac{\kappa}{\tilde{u}+\kappa-i\epsilon} \right] d\tilde{u} \\ &= 2 \int_{-i}^\infty \left[ \frac{1}{\tilde{u}-i\epsilon} - \frac{1}{\tilde{u}+\kappa-i\epsilon} \right] d\tilde{u} \\ &= 2 \left[ \ln \left( \frac{\tilde{u}-i\epsilon}{\tilde{u}+\kappa-i\epsilon} \right) \right]_{-i}^{+\infty} \\ &= -2 \left[ \ln \left( \frac{-i-i\epsilon}{-i+\kappa-i\epsilon} \right) \right] \\ &= +2 \ln(1+i\kappa). \end{aligned} \quad (48)$$

(The original contour was chosen to run underneath the two branch cuts emanating from  $u = -1 + i\epsilon$  and  $u = +1 + i\epsilon$ ; thus under the change of variables  $u'' = \sqrt{u^2-1}$  the branch cut must be chosen so that the new contour terminates at  $-i$  and not at  $+i$ .)

### The integral for acoustic junction conditions:

We are finally ready to study the case of interest: acoustic *junction conditions*. The reflection coefficient is now [20, equation (3.1.19), page 82]

$$R(\rho, n; u) = \frac{\rho\sqrt{1-u^2} - \sqrt{n^2-u^2}}{\rho\sqrt{1-u^2} + \sqrt{n^2-u^2}}. \quad (49)$$

*Consistency check I:* Note that  $\rho \rightarrow +\infty$  gives  $R = +1$ , as appropriate for Neumann and Robin boundary conditions;  $\rho \rightarrow 0$  gives  $R = -1$  as appropriate for the Dirichlet boundary condition; while  $\rho \rightarrow \infty$  with  $\kappa = -in/\rho$  fixed gives the surface damped boundary condition.

*Consistency check II:* Similarly  $n \rightarrow +\infty$  gives  $R = -1$ , as appropriate for Dirichlet boundary conditions; finally  $n \rightarrow +i\infty$  gives  $R = +1$  as appropriate for Neumann and Robin boundary conditions.

*Observation:* The reflection coefficient exhibits an inversion symmetry as we move from one side of the interface to the other, this symmetry being inherited by the  $\xi$ .

$$R_{in}(\rho, n; u) = R_{out}(1/\rho, 1/n; u). \quad (50)$$

Thus

$$\xi_{in}(\rho, n) = \xi_{out}(1/\rho, 1/n). \quad (51)$$

*Calculation:* We are interested in evaluating

$$Q = \text{Im} \left[ \int_0^\infty \frac{u du}{1-u^2} \frac{\rho\sqrt{1-u^2} - \sqrt{n^2-u^2}}{\rho\sqrt{1-u^2} + \sqrt{n^2-u^2}} \right]. \quad (52)$$

The integrand has a pole at  $u = 1$  of residue  $-1/2$ , and branch cuts emanating from  $u = \pm 1$  which can be chosen to terminate at  $u = \pm n$ . Asymptotically, as  $u \rightarrow \infty$ , the integrand goes as

$$\frac{1}{u} \frac{\rho-1}{\rho+1}. \quad (53)$$

This is already enough to tell us that the imaginary part of this integral can be finite if and only if  $\rho$  is real. For the acoustic equations this is actually very sensible physically since it is meaningless to drive the density complex. To evaluate this expression we subtract and add 1 to the integrand, and make use of the integral  $\int u du/(1-u^2)$ , evaluated in equation (43), to write

$$\begin{aligned} Q &= +\frac{\pi}{2} \\ &+ \text{Im} \left[ \int_0^\infty \frac{u du}{1-u^2} \left\{ \frac{\rho\sqrt{1-u^2} - \sqrt{n^2-u^2}}{\rho\sqrt{1-u^2} + \sqrt{n^2-u^2}} + 1 \right\} \right]. \end{aligned} \quad (54)$$

This conveniently gets rid of the pole so that the integral is now unambiguously finite. Indeed

$$Q = +\frac{\pi}{2} + 2\rho \text{Im} \left[ \int_0^\infty \frac{u du}{1-u^2} \frac{\sqrt{1-u^2}}{\rho\sqrt{1-u^2} + \sqrt{n^2-u^2}} \right]. \quad (55)$$

Now we also have to take  $n$  to be real, otherwise we step outside the Balian–Bloch formalism. For now, also take

$n > 1$ , the alternative case being completely analogous. The integrand is now imaginary only over the range  $u \in [1, n]$ , and we can change variables to set

$$Q = +\frac{\pi}{2} + \rho \left[ \int_1^{n^2} \frac{du'}{1-u'} \operatorname{Im} \left\{ \frac{i\sqrt{u'-1}}{\sqrt{n^2-u'} + i\rho\sqrt{u'-1}} \right\} \right]. \quad (56)$$

That is

$$Q = +\frac{\pi}{2} + \rho \left[ \int_1^{n^2} \frac{du'}{1-u'} \frac{\sqrt{n^2-u'}\sqrt{u'-1}}{(n^2-u') + \rho^2(u'-1)} \right]. \quad (57)$$

Equivalently

$$Q = +\frac{\pi}{2} - \rho \int_0^{n^2-1} \frac{du''}{u''} \frac{\sqrt{n^2-1-u''}\sqrt{u''}}{(n^2-1-u'') + \rho^2 u''}. \quad (58)$$

Now define  $u'' = (n^2 - 1)w$ .

$$Q = +\frac{\pi}{2} - \rho \int_0^1 \frac{dw}{w} \frac{\sqrt{1-w}\sqrt{w}}{1 + (\rho^2 - 1)w}. \quad (59)$$

Note that the refractive index  $n$  has now completely disappeared from the integral. This gives

$$Q = +\frac{\pi}{2} - \rho \frac{\pi}{\rho+1} = -\frac{\pi}{2} \left[ \frac{\rho-1}{\rho+1} \right]. \quad (60)$$

We can re-do the calculation for  $n < 1$ . A few intermediate steps change but the final result is the same. We finally have our announced result

$$\xi_{out}(\rho, n) = \frac{\pi}{4} \left[ \frac{\rho-1}{\rho+1} \right] = \frac{\pi}{4} \left[ \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} \right]. \quad (61)$$

Note the remarkable result that this is independent of  $n$  for  $n$  real. With hindsight, we can see that the acoustic junction conditions explicitly make reference only to the density of the fluid, and not to the velocity of sound (refractive index), which might be viewed as an *a posteriori* justification for the absence of refractive index in the final result. However we know of no simple physics argument that would justify this, and must rely on the explicit calculation presented above.

As  $\rho \rightarrow +\infty$  we recover Neumann and Robin boundary conditions while as  $\rho \rightarrow 0$  we recover Dirichlet boundary conditions. Also note that on interchanging the two regions,  $\rho \rightarrow 1/\rho$ , so we have

$$\xi_{in}(\rho) = \xi_{out}(1/\rho) = -\xi_{out}(\rho), \quad (62)$$

as expected from our earlier discussion [see Eq. (51)].

## C. Electromagnetic field

For the electromagnetic field, we can use the analysis presented by Balian and Bloch in [16, pages 273–274] to view the electromagnetic eigenvalue problem as a combination of vector and scalar eigenvalue problems. A standard result is

**Perfect conductor boundary conditions:**

( $\vec{E} \times \vec{n} = 0$  and  $\vec{B} \cdot \vec{n} = 0$  on the boundary)

$$\xi = 0. \quad (63)$$

This vanishing of the surface term for perfect conductor boundary conditions is due to a cancellation between TE and TM modes. (For a surface of general shape the separation into TE and TM modes is meaningless; TE and TM modes make sense only in situations of extremely high symmetry. Nevertheless, sufficiently close to any conducting surface we may approximate the surface by its tangent plane—and in this *approximation* the decomposition into TE and TM modes makes sense. The general vector minus scalar decomposition alluded to above then *approximately* reduces to the simpler scalar plus scalar decomposition for the TE and TM modes.)

**Dielectric junction conditions:**

For the case of ultimate interest we are of course interested in *dielectric junction conditions*. A full appreciation of the (perhaps unexpected) subtleties involved with dielectric junction conditions might be gleaned from the fact that even for a plane interface the situation is sufficiently complicated as to warrant a recent 600 page technical monograph [21], and a continuing stream of research papers (see for instance [22]).

Nevertheless we can make a few general statements on physical grounds before doing a detailed calculation of  $\xi$ . In analogy with the case of the scalar field, finite-volume effects will distort the density of states both *inside* and *outside* the dielectric body according to the general scheme

$$\sum_{inside} \sim V \int \frac{d^3 \vec{k}}{(2\pi)^3} + S \int \xi_{in} \frac{d^3 \vec{k}}{(2\pi)^3 k} + \dots \quad (64)$$

$$\sum_{outside} \sim (V_\infty - V) \int \frac{d^3 \vec{k}}{(2\pi)^3} + S \int \xi_{out} \frac{d^3 \vec{k}}{(2\pi)^3 k} + \dots \quad (65)$$

For the case of a dielectric junction, we expect  $\xi(\epsilon, \mu)$  to be a function of the permeability and permittivity, and we know, from first principles, that as  $\epsilon \rightarrow 1$  and  $\mu \rightarrow 1$  the dielectric boundary disappears as both media become the same, so we must have

$$\xi(\epsilon, \mu) \rightarrow 0 \quad \text{as} \quad \epsilon \rightarrow 1 \quad \text{and} \quad \mu \rightarrow 1. \quad (66)$$

When we turn to including dispersive effects we note that  $\xi(\epsilon, \mu)$  should ultimately be taken to be a function of the



wave-number dependent quantities  $\epsilon(k)$ ,  $\mu(k)$ . Since we know that as  $k \rightarrow \infty$  the dielectric must ultimately mimic individual atoms embedded in vacuum, we must have

$$\xi(\epsilon(k), \mu(k)) \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty. \quad (67)$$

The calculation of  $\xi$  for the electromagnetic field is an easy exercise given our results for the acoustic problem. We decompose the electromagnetic field near the approximately plane boundary into TE and TM modes. In terms of the relative refractive index, relative permittivity, and relative permeability, the reflection coefficients (for the outside region) are simply [20, pages 83–84] (or see [23, equations (86.4) and (86.6), page 295], or [24, pages 281–282])<sup>2</sup>

$$\begin{aligned} R^{TE}(\epsilon, \mu; u) &= R^{acoustic}(\rho = \mu, n; u) \\ &= \frac{\mu\sqrt{1-u^2} - \sqrt{n^2-u^2}}{\mu\sqrt{1-u^2} + \sqrt{n^2-u^2}}. \end{aligned} \quad (68)$$

$$\begin{aligned} R^{TM}(\epsilon, \mu; u) &= R^{acoustic}(\rho = \epsilon, n; u) \\ &= \frac{\epsilon\sqrt{1-u^2} - \sqrt{n^2-u^2}}{\epsilon\sqrt{1-u^2} + \sqrt{n^2-u^2}}. \end{aligned} \quad (69)$$

(Remember that  $n = \sqrt{\epsilon\mu}$ . Also, we have defined  $n = n_1/n_2$ ,  $\epsilon = \epsilon_1/\epsilon_2$ , and  $\mu = \mu_1/\mu_2$ )

Thus, applying the previous acoustic results, we get the remarkably simple formulae:

$$\xi_{out}^{TE}(\mu) = \frac{\pi}{4} \left[ \frac{\mu-1}{\mu+1} \right] = \frac{\pi}{4} \left[ \frac{\mu_1-\mu_2}{\mu_1+\mu_2} \right] = -\xi_{in}^{TE}(\mu). \quad (70)$$

$$\xi_{out}^{TM}(\epsilon) = \frac{\pi}{4} \left[ \frac{\epsilon-1}{\epsilon+1} \right] = \frac{\pi}{4} \left[ \frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2} \right] = -\xi_{in}^{TM}(\epsilon). \quad (71)$$

Note that the result for the TE modes is independent of  $\epsilon$ , while that for the TM mode is independent of  $\mu$ . Since most typical dielectric materials are magnetically inert,  $\mu \approx 1$ , the TE contribution is typically much smaller than the TM contribution.

*Consistency check:* Instead of appealing to the identification of reflection coefficients, we can get the same results directly from the dielectric boundary conditions. We know that

$$\vec{E}^\perp, \quad \epsilon \vec{E}^n, \quad \vec{H}^\perp, \quad \text{and} \quad \mu \vec{H}^n, \quad (72)$$

must be continuous across the boundary.

If we are dealing with a plane interface, or in the approximation that we are sufficiently close to a curved interface, specifying the normal components of the  $\vec{E}$  and

$\vec{B}$  fields is sufficient to completely determine the electromagnetic field. In terms of these normal components the junction conditions are simply

*TE mode:*

$$\mu_1 H_1^n = \mu_2 H_2^n, \quad (73)$$

$$\partial_n H_1^n = \partial_n H_2^n. \quad (74)$$

*TM mode:*

$$\epsilon_1 E_1^n = \epsilon_2 E_2^n, \quad (75)$$

$$\partial_n E_1^n = \partial_n E_2^n. \quad (76)$$

Applying the formalism derived for the acoustic junction conditions, the previously quoted results for  $\xi$  immediately follow.

#### IV. THE CASIMIR ENERGY

Including these surface contributions to the density of states, the total zero-point energy for a dielectric body embedded in a background dielectric is easily seen to be

$$\begin{aligned} E_{embedded-body} &= 2V \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{1}{2} \hbar \omega_1(k) \\ &\quad + 2S \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{1}{2} \hbar c \left[ \bar{\xi}_{in}(\epsilon, \mu) \frac{\omega_1(k)}{ck} \right] \\ &\quad + 2(V_\infty - V) \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{1}{2} \hbar \omega_2(k) \\ &\quad + 2S \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{1}{2} \hbar c \left[ \bar{\xi}_{out}(\epsilon, \mu) \frac{\omega_2(k)}{ck} \right] \\ &\quad + \dots \end{aligned} \quad (77)$$

This is just the generalization of equation (4) above to include surface effects. The quantity  $\bar{\xi}$  denotes an *average* over TE and TM modes. To calculate the Casimir energy we now simply subtract the homogeneous dielectric zero-point energy [equation (7)] to obtain

$$\begin{aligned} E_{Casimir} &= 2V \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{1}{2} \hbar [\omega_1(k) - \omega_2(k)] \\ &\quad + 2S \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{1}{2} \hbar c \left[ \frac{\bar{\xi}_{in}(\epsilon, \mu)}{n_1} + \frac{\bar{\xi}_{out}(\epsilon, \mu)}{n_2} \right] \\ &\quad + \dots \end{aligned} \quad (78)$$

Even though the surface terms seem to be additive, there is a “hidden” minus sign, as we shall see below, due to the fact that  $\xi_{in}(\rho) = -\xi_{out}(\rho)$ .

This is quite enough to give a good qualitative feel for the physics: the Casimir effect will in general induce a surface tension that goes as (cutoff)<sup>3</sup>.

<sup>2</sup>Be careful with all the different notations in use.

It is useful to define

$$\Xi(\epsilon_1, \mu_1; \epsilon_2, \mu_2) = \left[ \frac{\bar{\xi}_{in}(\epsilon, \mu)}{n_1} + \frac{\bar{\xi}_{out}(\epsilon, \mu)}{n_2} \right] \quad (79)$$

and so write the Casimir surface tension as

$$\sigma(\text{surface tension}) = \int \frac{d^3 \vec{k}}{(2\pi)^3} \hbar c \Xi(\epsilon_1, \mu_1; \epsilon_2, \mu_2). \quad (80)$$

From our previous results for  $\xi$ , taking the case of magnetically inert media for simplicity ( $\mu = 1$ ), we see

$$\Xi(n_1, n_2) = \frac{\pi}{8} \left[ -\frac{1}{n_1} + \frac{1}{n_2} \right] \frac{n_1^2 - n_2^2}{n_1^2 + n_2^2}. \quad (81)$$

Here we indeed see that the two surface terms contribute with opposite signs, largely cancelling each other. We can factorize this to yield

$$\Xi(n_1, n_2) = +\frac{\pi}{8} \frac{(n_1 - n_2)^2 (n_1 + n_2)}{n_1 n_2 (n_1^2 + n_2^2)}. \quad (82)$$

Note that this vanishes as  $(n_1 - n_2)^2$ , with one factor of  $(n_1 - n_2)$  coming from the fact that the  $\xi_i$  individually tend to zero as  $n_1 \rightarrow n_2$  and the second coming from the partial cancellation discussed above.

What does this do to the Casimir energy?

$$\begin{aligned} E_{Casimir} &= 2V \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{1}{2} \hbar c k \left[ \frac{n_2 - n_1}{n_1 n_2} \right] \\ &+ 2S \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{1}{2} \hbar c \frac{\pi}{8} \frac{(n_1 - n_2)^2 (n_1 + n_2)}{n_1 n_2 (n_1^2 + n_2^2)} \\ &+ \dots \end{aligned} \quad (83)$$

This is our general result for the Casimir energy. We now insert a momentum dependent refractive index into the above to explicitly evaluate the coefficients. The physical cutoff is provided by the fact that both refractive indices are known to tend to 1 at large momenta.

A naive hard cutoff, following the ideas of Schwinger, simplifies these expressions considerably. Naive hard cutoffs are of course an idealization that suppresses much of the physical detail, and are justified only for order of magnitude estimates and for comparison with the previous literature where naive hard cutoffs are often the only extant results. Suppose we take

$$n_1(k) = n_1 \Theta(K - k) + \Theta(k - K), \quad (84)$$

and

$$n_2(k) = n_2 \Theta(K - k) + \Theta(k - K). \quad (85)$$

(It is an additional gross over-simplification to set the cutoffs for the two media equal to one another, but it is standard and is the only way to make connection with previous calculations. Keeping separate cutoffs for the

two media is straightforward but algebraically somewhat messy.)

The Casimir energy is then given by

$$\begin{aligned} E_{Casimir} &= \frac{1}{8\pi^2} V \hbar c K^4 \left[ \frac{1}{n_1} - \frac{1}{n_2} \right] \\ &+ \frac{1}{6\pi^2} S \hbar c K^3 \left[ \frac{\bar{\xi}_{in}(n_1, n_2)}{n_1} + \frac{\bar{\xi}_{out}(n_1, n_2)}{n_2} \right] \\ &+ \dots, \end{aligned} \quad (86)$$

while the Casimir surface tension is

$$\sigma = \frac{1}{6\pi^2} \hbar c K^3 \left[ \frac{\bar{\xi}_{in}(n_1, n_2)}{n_1} + \frac{\bar{\xi}_{out}(n_1, n_2)}{n_2} \right]. \quad (87)$$

Inserting the specific formulae for  $\xi$  then yields

$$\sigma = +\frac{1}{48\pi} \hbar c \frac{(n_1 - n_2)^2 (n_1 + n_2)}{n_1 n_2 (n_1^2 + n_2^2)} K^3. \quad (88)$$

Now particularize to dilute media, by taking  $n_1 \approx 1 \approx n_2$ .

$$\begin{aligned} E_{Casimir} &\approx \frac{1}{8\pi^2} V \hbar c K^4 [n_2 - n_1] \\ &+ \frac{1}{48\pi} S \hbar c K^3 [(n_1 - n_2)^2] + \dots \end{aligned} \quad (89)$$

The volume term here is the dilute medium limit of Schwinger's result [1], while the surface area term reproduces the Milton *et al.* result [12–14]. There is an overall normalization difference between this surface term and the special case calculated by Milton *et al.*, this normalization difference being attributable to a different choice of regulator. The critical physics lies in the volume versus surface area dependence, the power of the cutoff dependence, and the behaviour as a function of refractive index.

Note that the bulk term is dominant if

$$V K \gg S, \quad (90)$$

that is, for dielectrics with linear dimensions satisfying

$$L \sim (V/S) \gg 1/K = \lambda_0/(2\pi). \quad (91)$$

For a typical dielectric we estimate  $\lambda_0 \approx 1,000$  Angstrom, so for dielectrics of this size or greater the Casimir energy will be dominated by bulk effect. *This is certainly the case for sonoluminescence where typical bubble radii are of order 100,000 Angstroms.*

For small enough dielectric particles the surface term will not be negligible in comparison to the volume term—this is no great surprise to people studying mesoscopic systems for which the existence of finite volume effects is well known.

Finally, we mention that for [non-dispersive] Neumann, Dirichlet, and Robin boundary conditions the existence of a surface term contributing to the total Casimir energy has been known for some time—see for instance [25].

## V. DISCUSSION

The main results of this paper are:

- (1) The Casimir energy in a dielectric medium is dominated by a volume term. Indeed, for a finite-volume of dielectric 1, embedded in an infinite volume of different dielectric 2,

$$E_{Casimir} = 2V \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2} \hbar [\omega_1(k) - \omega_2(k)] \\ + 2S \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2} \hbar c \Xi(\epsilon_1, \mu_1; \epsilon_2, \mu_2) \\ + \dots, \quad (92)$$

where the dots represent terms arising from higher-order distortions of the density of states due to finite-volume effects.

- (2) If we adopt a simple cutoff model for the dispersion relation, the volume term is

$$E_{Casimir}^{bulk} = \frac{1}{8\pi^2} V \hbar c K^4 \left[ \frac{1}{n_1} - \frac{1}{n_2} \right]. \quad (93)$$

This result is completely in agreement with Schwinger's calculation in [1], and in disagreement with [13,14].

- (3) In addition, there will be a sub-dominant contribution to the Casimir energy that is proportional to the surface area of the dielectric. This surface contribution takes the generic form

$$E_{Casimir}^{surface} = + \frac{1}{48\pi} S \hbar c K^3 \left[ \frac{(n_1 - n_2)^2 (n_1 + n_2)}{n_1 n_2 (n_1^2 + n_2^2)} \right]. \quad (94)$$

This term is sub-dominant provided

$$V/S \gg 1/K = \lambda_0/(2\pi). \quad (95)$$

- (4) In general, we can expect these to be the first two terms of a more general expansion that includes terms proportional to various geometrical invariants of the body. By analogy with the situation for non-dispersive Dirichlet, Neumann, and Robin boundary conditions [15] we expect the next term to be proportional to the trace of the extrinsic curvature integrated over the surface of the body.

- (5) The analysis of the present paper has been limited to situations of real refractive index (loss-free insulating dielectrics). Generalizing to lossy conducting media is clearly of interest but will require a careful re-assessment of the entire formalism.

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